Practical application of Moment of Inertia (1)

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Moment of Inertia, un-veiling its character

Recently there have been several articles in Guild Journal about Moment of Inertia, (MoI). The basic concept and idea about MoI as it applies to pianos was well presented in those articles.

I have also been thinking for a while about how to use the MoI in piano work and I hope that my thoughts about its practical application add to the discussion.

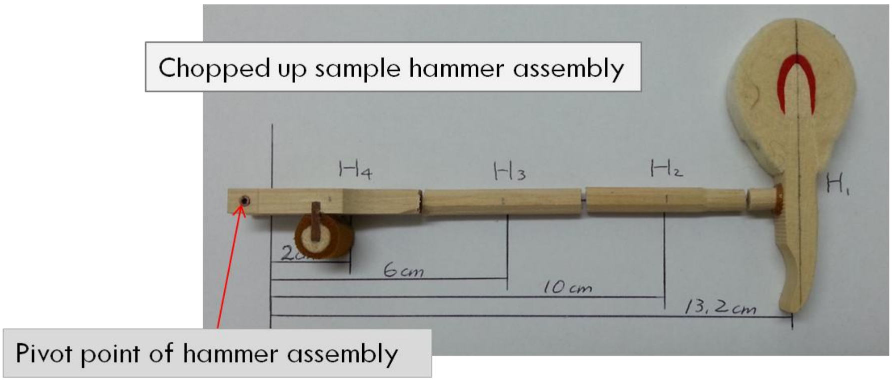
My basic method of calculating the MoI in a piano mechanism is to assign MoI value to each part or section of the three parts, hammer assembly, whippen and key stick. MoI theory would suggest this is the correct method.

Moment of Inertia of a body is calculated by summing mr^2 (mass x radius^2) for every particle in the body about a given rotation axis.

I have taken a simplified practical approach and separated each part into sections of a known size. Then the mass of each section, multiplied by the squared distance from the pivot point is added together to get a MoI value. An example will help.

Measuring Moment of Inertia of action parts

The Hammer assembly



(Fig. 1) Hammer assembly separated into 4 parts to calculate its MoI

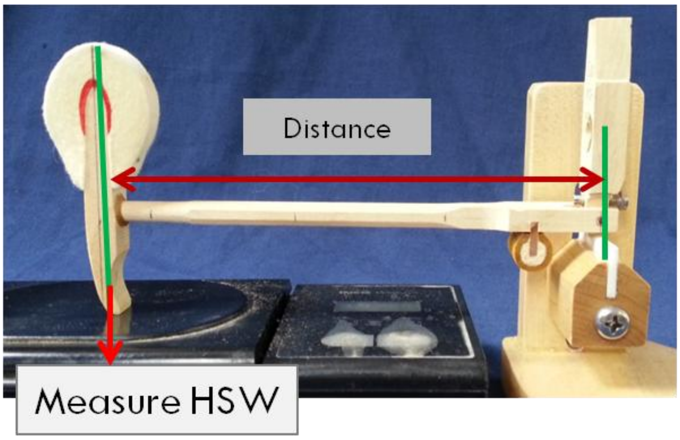
The sample hammer was divided into 4 parts as seen at Fig. 1. Theory suggests each part should be as small as possible, but cutting it every 4 cm, as in the photo will still be sufficient for our analysis. The flange is not included because it is stationary.

The mass of each part is H_1, H_2, H_3 and H_4. The MoI for this hammer is then calculated by:

$$\textbf{Moment of Inertia}_{\text{(Hammer)}} = H_1 \times (2)^2 + H_2 \times (6)^2 + H_3 \times (10)^2 + H_4 \times (13.2)^2$$

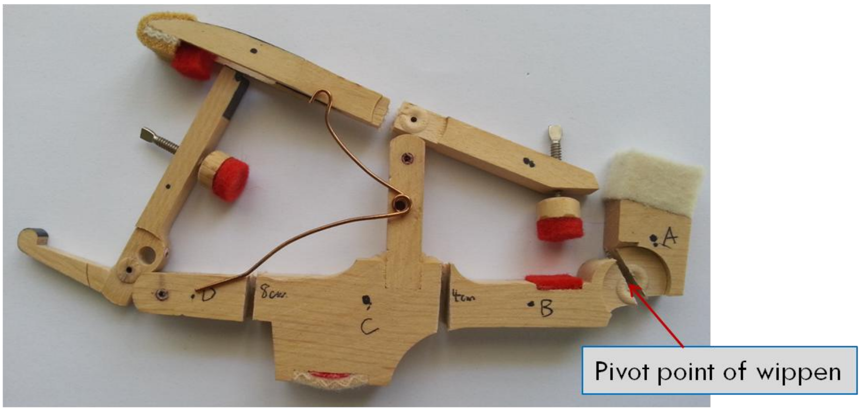
Weighing the component parts, the calculated MoI of this sample hammer is 1616 gcm^2. (Note I am using gcm^2 instead of Kgm^2 which is normally used for MoI calculations simply because the gcm^2 values better fit our purpose.)

Cutting up hammer shanks to calculate their MoI is however impractical. So I suggest an alternative method. The hammer strike weight multiplied by the square of the distance between pivot point and centre line of the hammer moulding (Fig. 2). For example, the sample hammer before we chopped it up, it calculated 1,720 gcm^2. (Comment: this is about 6.5 % different from the same sample segmented). This method works well because the hammer strike weight needs to be measured to analyze the action using the Stanwood method and the hammer head has the majority of mass within hammer assembly. So this is a very practical method of calculating the MoI of hammer assemblies.



(Fig. 2) Measuring the hammer strike weight using the Stanwood system to get a MoI value

Calculating the Moment of Inertia for wippen assemblies



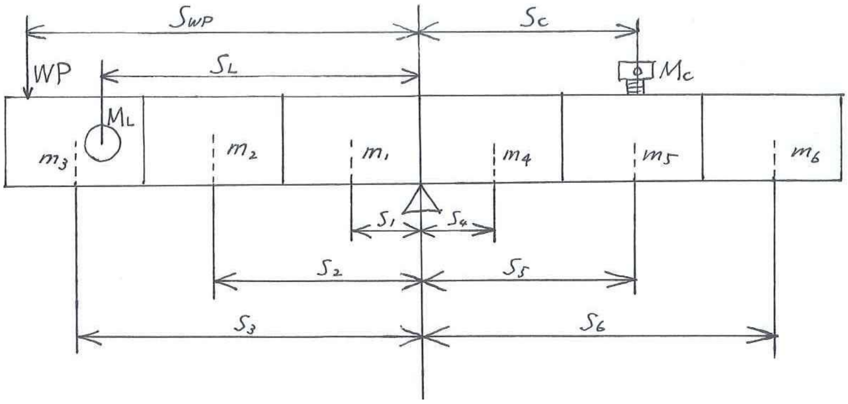
(Fig. 3) Segmented wippen for measuring MoI

Our sample wippen was separated into 8 parts (Fig. 3). The flange is not included, as like the hammer flange, it is stationary. The MoI of the wippen can then be calculated by the equation below.

Moment of Inertia $_{(Whippen)}$ = $\boldsymbol{\Sigma m_n(s_n)^2}$ (Σ is the sum of segments, where m is mass, s is the distance between the mass center and pivot point and n is each part)

The MoI of the wippen above is then calculated at 756 gcm^2. Later we are going to compare the MoI of the three action components, hammer assembly, whippen, key stick, but already we can see that whippen has a much lower MoI value than the hammer assembly. So it will have a small effect on the total MoI.

The Key: The MoI of the key is best explained by Fig. 4.



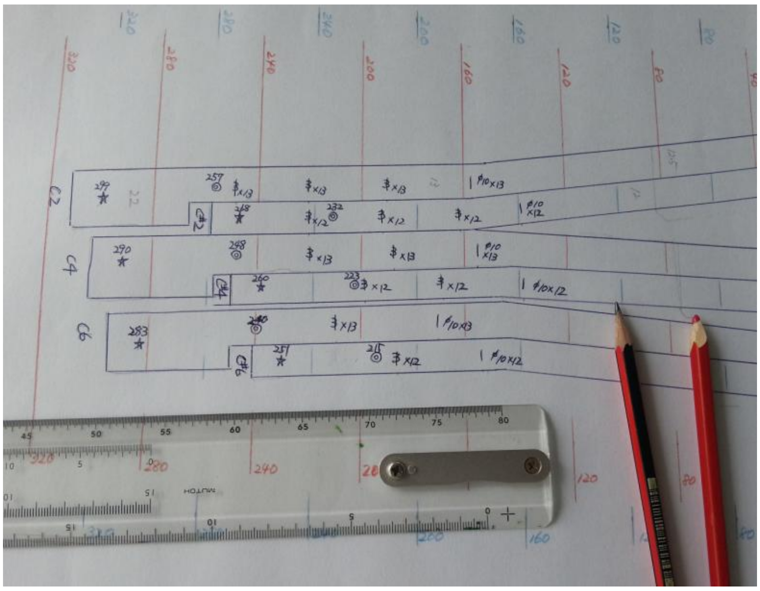
(Fig. 4) Simplified model of key stick explains measuring the MoI.

The MoI for the key is calculated using the same method used for hammer assembly and wippen. The MoI for the key is again calculated by adding the values for each of the parts. (The 4cm division is an arbitrary but practical division).

The MoI of the above sketch model (Fig. 4) would be calculated:

Moment of Inertia $_{(key)}$ = $\mathbf{m_1(s_1)^2 + m_2(s_2)^2 + m_3(s_3)^2 + m_L(s_L)^2 + m_4(s_4)^2 + m_5(s_5)^2 + m_6(s_6)^2 + m_c(s_c)^2}$

For the actual measurement, a simple method is to make a paper template, as in Fig. 5 from which we extract the details we need.



(Fig 5) Making a template for measuring the MoI of keys

Draw the outline of the keys, mark the position of the leads and record the lead diameter and depth, the position of the balance rail pin, the capstan and back check, record the thickness and width of the key stick, record where any dense timber is used, (back-check block etc.) (Fig. 5). To record the capstan weight, you will need to remove one. To record other parts, the easiest way is to use spares you may have in your shop e.g. ebony keys, plastic key tops and spare back checks, measure and use their data. The mass of the spruce and heavier timbers can be calculated by multiplying their volume by their density. You need to calculate the volume of the leads by using $\pi r^2 \times h$ (in this case h is the depth of the lead).

All measured data is then put into my MoI spread sheet, (see Fig. 6) MoI values then calculate automatically.

Nakamura Moment of Inertia (key) spread sheet

	Key #	Note	Status	Whole Inertia	FW point dstnc	Center of torqu	CoG partition ratio	Lead (front) distance #5	mass of lead	distance #4	mass of lead	distance #3	mass of lead	distance #2	mass of lead	distance #1	mass of lead
A	40	C4	Original	50403	292	222.8	0.731	244	14.5	220	8.6	157	4.3				
B			leastinertia	41181	292	84.7	0.278					119	19.4	85	19.4	50	19.4
C			practicalsettin	44657	292	152.8	0.501			220	8.6	157	4.3	122	19.4		
D			keystick only	36548	292												

	Wood (front) (mm) 360~320	320~280	280~240	240~200	200~160	160~120	120~80	80~40	40~0	Key top distance (mm)	mass (g)	Wood (back) (mm) 0~40	40~80	80~120	120~160	160~200	200~240	240~280	280~320	320~360
A		4.5	6.2	4.2	4.4	4.6	4.6	5.3	6.7	280	7.5	7.7	5.7	5.7	4.6	3.6	3.4	4.1	0.5	
B		4.5	6.2	4.2	4.4	4.6	4.6	5.3	6.7	280	7.5	7.7	5.7	5.7	4.6	3.6	3.4	4.1	0.5	
C		4.5	6.2	4.2	4.4	4.6	4.6	5.3	6.7	280	7.5	7.7	5.7	5.7	4.6	3.6	3.4	4.1	0.5	
D		4.5	6.2	4.2	4.4	4.6	4.6	5.3	6.7	280	7.5	7.7	5.7	5.7	4.6	3.6	3.4	4.1	0.5	

	Capstn distance capstan (mm)	capstan weight	check distance check (mm)	check weight (g)	Lead (back) distance #1 (mm)	mass (g)	distance #2 (mm)	mass (g)	Moment of Inertia front	back	Whole key g cm2
A	151.0	6.7	249.0	5.2					37910	12493	50403
B	151.0	6.7	249.0	5.2					28689	12493	41181
C	151.0	6.7	249.0	5.2					32164	12493	44657
D	151.0	6.7	249.0	5.2					24055	12493	36548

(Fig. 6) Nakamura MoI (key) spread sheet.

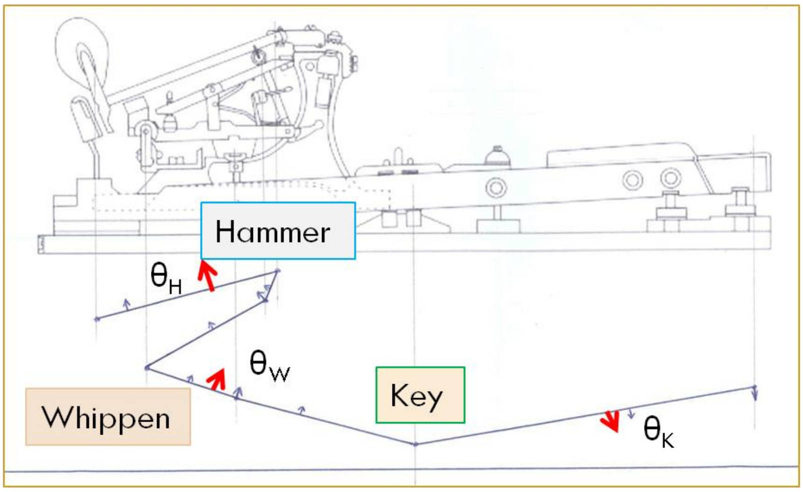
Perhaps it will be helpful to give some sample values from data I have collected.

- The MoI of key stick, A0 of a Steinway D: 72,000 gcm^2
- The MoI of key stick, C4 of a Steinway D: 50,400 gcm^2
- The MoI of key stick, C4 of Yamaha C3: 31,000 gcm^2
- The MoI of key stick, C4 of Kawai K3: 6,000 gcm^2.

There are huge differences between manufacturers, grands and uprights.

Calculating whole action Moment of Inertia at the key

We feel the MoI of the whole action where we play the key. The article by Fandrich and Rhodes in the July 2013 issue, explains the compounding MoI effect of the ratio of angles in a piano mechanism.



(Fig. 7) Angle of θ_K at the front of key is transferred to θ_W at wippen and θ_H at hammer

When the key is depressed by the small angle θ_K, the whippen rotates θ_W and the hammer rotates θ_H (Fig. 7).

According to theory, the MoI of a chain of objects can be found by multiplying the MoI of the second linked object and the ratio of rotation between the two objects. So the total amount of MoI felt at the front of the key can be written:

$$\mathbf{MoI}_{(\text{Whole action at key})} = \mathbf{MoI}_{(K)} + \mathbf{MoI}_{(W\text{ at key})} + \mathbf{MoI}_{(H\text{ at key})}$$

$$= \mathbf{MoI}_{(K)} + \mathbf{MoI}_{(W)} \times (\theta_W/\theta_K)^2 + \mathbf{MoI}_{(H)} \times (\theta_H/\theta_K)^2$$

We can re-write this equation as a ratio of lengths, as follows: (lengths are more easily measured).

$$\mathbf{MoI}_{(\text{Whole action at key})} = \mathbf{MoI}_{(K)} + \mathbf{MoI}_{(W\text{ at key})} + \mathbf{MoI}_{(H\text{ at key})}$$

$$= \mathbf{MoI}_{(K)} + \mathbf{MoI}_{(W)} \times (L_{KO/}/ L_{WI})^2 + \mathbf{MoI}_{(H)} \times (L_{WO/}/ L_{HI} \times L_{KO/}/ L_{WI})^2$$

L_{KO} is the distance between the pivot of the key and the top of the capstan screw. L_{WI} is distance between the pivot of the wippen and the center bottom of the wippen heel. We can then calculate the amount of MoI transferred to the key, as below.

When the capstan moves $L_{KO} \times \theta_K$, the wippen heel is moved $L_{WI} \times \theta_{W,}$ as both surfaces move together, the actual travel is the same amount.

So we can write the equation: $\mathbf{L_{KO} \times \theta_K = L_{WI} \times \theta_W}$

Rearranging the equation, we get $\mathbf{\theta_W/\theta_K = L_{KO/}/ L_{WI}}$

The MoI of the wippen, felt at the key can then be written:

$$\mathbf{MoI}_{(W\text{ at key})} = \mathbf{MoI}_{(W)} \times (\theta_W/\theta_K)^2 = \mathbf{MoI}_{(W)} \times (L_{KO/}/ L_{WI})^2$$

Therefore we can get the value of the MoI of the whippen felt at the key simply by measuring two lengths, $\mathbf{L_{KO}}$ and $\mathbf{L_{WI}}$ as above.

Apply the same principle to the hammer.

The MoI of the hammer, felt at the key front is:

$$\mathbf{MoI}_{(H\text{ at key})} = \mathbf{MoI}_{(H)} \times (\theta_H/\theta_K)^2 = \mathbf{MoI}_{(H)} \times (\theta_H/\theta_W \times \theta_W/\theta_K)^2$$

θ_H/θ_W is found the same way as above. L_{WO} is the distance between pivot of wippen (wippen center pin) and the contact point, jack to hammer knuckle and L_{HI} is the distance between contact point, knuckle to jack and shank flange center.

When the jack moves $L_{WO} \times \theta_{W,}$ the knuckle is moved $L_{HI} \times \theta_H$ as both surfaces move together, the actual travel is the same amount.

So we can write the equation: $\mathbf{L_{WO} \times \theta_W = L_{HI} \times \theta_H}$

Rearranging the equation, we get: $\mathbf{\theta_H/\theta_W = L_{WO/}/ L_H}$

The MoI of the hammer, (excluding the MoI of key and whippen) felt at front of the key can be written:

$\mathbf{MoI_{(H\ at\ key)} = MoI_{(H)} \times (\theta_H/\theta_K)^2 = MoI_{(H)} \times (\theta_H/\theta_W \times \theta_W/\theta_K)^2 = MoI_{(H)} \times (L_{WO}/\ L_{HI} \times L_{KO}/\ L_{WI})^2}$

So the MoI of the whole action, felt at key is written:

$$\mathbf{MoI_{(Whole\ action\ at\ key)} = MoI_{(K)} + MoI_{(W\ at\ key)} + MoI_{(H\ at\ key)}}$$

$$\mathbf{= MoI_{(K)} + MoI_{(W)} \times (L_{KO}/L_{WI})^2 + MoI_{(H)} \times (L_{WO}/L_{HI} \times L_{KO}/L_{WI})^2}$$

We can therefore calculate the MoI of the whole action, felt at the key by using 4 measurable lengths.

1. $\mathbf{L_{WO}}$: the distance between pivot of the wippen and the contact point, jack to hammer knuckle

2. $\mathbf{L_{HI}}$: the distance between contact point, the knuckle to jack and the pivot of the shank flange

3. $\mathbf{L_{KO}}$: the distance between the pivot of the key and the top of the capstan screw

4. $\mathbf{L_{WI}}$: the distance between the pivot of the wippen and the center bottom of the wippen heel

For example, here is data taken from C4 of a Steinway D.

The MoI of the hammer itself was HSW x L^2, which was 10.9g x 13^2. And (θ_H/θ_K) was 10.5, so the MoI of the hammer, felt at the key was 202,577 gcm^2.

The MoI of the wippen was 756 gcm^2 and (θ_w/θ_K) which was 2.14, so the MoI of the wippen, felt at the key was 4,332 gcm^2.

The MoI of the key was calculated at 50,463 gcm^2.

So adding these three figures gives us the MoI for this note of 257,311 gcm^2.

In part 2, I will show how to adjust the MoI of the action at same time as adjusting balance weight. It is possible to calculate them together without actual modification or complicating calculation by using proper several spread sheets.